Signals and Systems

Lecture 18

Fourier Transform Examples:

① $x(t) = 1$

Here $\int_{-\infty}^{\infty} |x(t)|\,dt = \int_{-\infty}^{\infty} dt \to \infty$. This means Dirichlet condition is not satisfied, But the Fourier Transform can be Calculated using Duality property.

$$\delta(t) \xleftrightarrow{FT} 1$$

$$X(\omega) = 1$$

$$\delta(t) \longleftrightarrow X(\omega)$$

using duality property:

$$X(t) \leftrightarrow 2\pi\, x(-\omega)$$

Here $X(t) = 1$, then $x(-\omega) = \delta(-\omega) \Rightarrow$

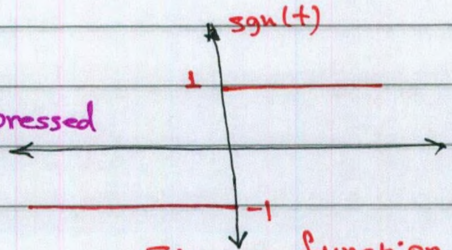
$$1 \longleftrightarrow 2\pi\,\delta(-\omega)$$

$\delta(\omega) \to$ is even functions of ω, $\overset{so}{\Rightarrow}$ $\delta(\omega) = \delta(-\omega)$, then

$$\boxed{1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)}$$

② $x(t) = \text{sgn}(t)$

The sgn(t) function can be expressed as: $x(t) = 2u(t) - 1$



Signum function

Differentiating both the sides,

$$\therefore \quad \frac{dx(t)}{dt} = 2\frac{d}{dt}u(t) = 2\,\delta(t)$$

Taking FT of both sides:

$$F\left[\frac{d}{dt}x(t)\right] = 2F[\delta(t)]$$

$$j\omega\, X(\omega) = 2 \Rightarrow X(\omega) = \frac{2}{j\omega}$$

$$\boxed{\text{Thus} \quad \text{sgn}(t) \xleftrightarrow{FT} \frac{2}{j\omega}}$$

③ $x(t) = u(t)$

Using example 2 $\Rightarrow$

$$\text{sgn}(t) = 2u(t) - 1 \Rightarrow 2u(t) = 1 + \text{sgn}(t)$$

Taking FT of both sides:

$$F[2u(t)] = F[1] + F[\text{sgn}(t)]$$

$$2u(t) \longleftrightarrow 2\pi\,\delta(\omega) + \frac{2}{j\omega}$$

$$\boxed{u(t) \xleftrightarrow{FT} \pi\,\delta(\omega) + \frac{1}{j\omega}}$$

* Note: We ~~Can't~~ Can't Calculate $F\{u(t)\}$ using equation directly. Dirichlet conditions not satified

4) $x(t) = \cos(\omega_0 t)$

$$x(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

We know that,

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$\frac{1}{2} \longleftrightarrow \pi\delta(\omega)$$

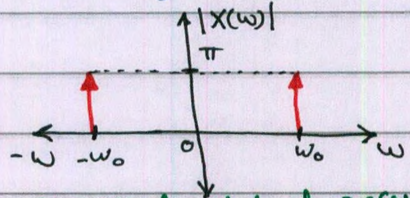
Using shifting property: $\left(e^{j\omega_0 t}x(t) \xleftrightarrow{FT} X(\omega-\omega_0)\right)$, then

$$\frac{1}{2}e^{j\omega_0 t} \xleftrightarrow{FT} \pi\delta(\omega-\omega_0)$$

Similarly $\frac{1}{2}e^{-j\omega_0 t} \xleftrightarrow{FT} \pi\delta(\omega+\omega_0)$

$$\Rightarrow F[x(t)] = FT\left\{\frac{1}{2}\cdot e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right] = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

$$\boxed{\cos(\omega_0 t) \xleftrightarrow{FT} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]}$$



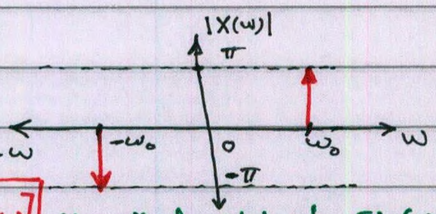
Magnitude plot of $\cos(\omega_0 t)$

5) $x(t) = \sin(\omega_0 t)$

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$$X(\omega) = \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)$$

Thus, $\boxed{\sin(\omega_0 t) \xleftrightarrow{FT} \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]}$



Magnitude plot of $\sin(\omega_0 t)$

6) $x(t) = u(-t)$

Using reversal property:

$$x(-t) = X(-\omega)$$

$$F\{u(-t)\} = \pi\delta(\omega) - \frac{1}{j\omega}$$

$$\boxed{u(-t) \xleftrightarrow{FT} \pi\delta(\omega) - \frac{1}{j\omega}}$$

Note: Dirichlet conditions not satisfied for $x(t) = u(-t)$

7) $x(t) = e^{j\omega_0 t}$

Let $y(t) = 1 \Rightarrow Y(j\omega) = 2\pi\delta(\omega)$, By using the frequency shifting property, we get

$$X(j\omega) = 2\pi\delta(\omega-\omega_0) \Rightarrow \boxed{e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)}$$

8) $x(t) = e^{-j\omega_0 t}$

$x(t) = e^{-j\omega_0 t} = e^{-j\omega_0 t}\cdot 1 \rightarrow$ using frequency shifting property:

$$\boxed{e^{-j\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega+\omega_0)}$$

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⑨ $x(t) = \delta(t-2)$

The impulse is time shifted by $t_0 = 2$

$$F[\delta(t-2)] = e^{-j\omega_0 t} F[\delta(t)] = e^{-j2\omega}$$

$$F[\delta(t-2)] = e^{-j2\omega}$$

⑩ $x(t) = \delta(t-1) - \delta(t+1)$

$$F[\delta(t-1)] = e^{-j\omega}$$

$$F[\delta(t+1)] = e^{j\omega}$$

$$F[\delta(t-1) + \delta(t+1)] = e^{-j\omega} - e^{j\omega} = -2j\sin(\omega)$$

$$F[\delta(t-1) + \delta(t+1)] = -2j\sin(\omega)$$

⑪ $x(t) = t \cdot e^{-at} \cdot u(t)$

$$F[e^{-at} \cdot u(t)] = \frac{1}{a+j\omega}$$

Using the FT property of differentiation in frequency, we get :

$$F[t \cdot e^{-at} \cdot u(t)] = j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right] = \frac{j(-j)}{(a+j\omega)^2}$$

$$X(\omega) = \frac{1}{(a+j\omega)^2}$$

$$\boxed{t \cdot e^{-at} \cdot u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^2}}$$

⑫ $x(t) = \cos(\omega_0 t + \theta)$

$$\cos(\omega_0 t + \theta) = \frac{1}{2}\left[e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}\right] = \frac{1}{2}\left[e^{j\theta} \cdot e^{j\omega_0 t} + e^{-j\theta} \cdot e^{-j\omega_0 t}\right]$$

By shifting property in frequency domain

$$F[e^{j\omega_0 t}] = 2\pi\,\delta(\omega-\omega_0)$$

$$F[e^{-j\omega_0 t}] = 2\pi\,\delta(\omega+\omega_0)$$

$$F[x(t)] = X(j\omega) = \frac{2\pi}{2}\left[e^{j\theta}\delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\right]$$

$$X(\omega) = \pi\left[e^{j\theta} \cdot \delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\right]$$

⑬ $x(t) = \cos(6\pi t - \frac{\pi}{8})$:

Using Example 12 :

$\omega_0 = 6\pi$ and $\theta = -\pi/8$

$$F[\cos(\omega_0 t + \theta)] = \pi\left[e^{j\theta}\delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\right]$$

$$F[\cos(6\pi t - \tfrac{\pi}{8})] = \pi\left[e^{-j\frac{\pi}{8}}\delta(\omega - 6\pi) + e^{j\frac{\pi}{8}} \cdot \delta(\omega + 6\pi)\right]$$